PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Part I. The Diffraction Theory of Sea Waves and the Shelter Afforded by Breakwaters

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PART I. THE DIFFRACTION THEORY OF SEA WAVES AND THE SHELTER AFFORDED BY BREAKWATERS

By W. G. PENNEY, F.R.S. and A. T. PRICE

CONTENTS

The diffraction of sea waves round the end of a long straight breakwater is investigated, use being made of the solutions of mathematically analogous problems in the diffraction of light. The wave patterns and wave heights are determined on both the leeward and windward sides of the breakwater, and for points quite close to the breakwater. This involves some extension of the calculations previously made for optical phenomena. The conditions obtaining in the lee of a small island are discussed. The penetration of waves through a single gap in a long breakwater is examined, and the result is shown to depend very much on whether the gap is small or not compared with the length of the waves. The investigation was suggested by problems arising in the construction of the Mulberry harbours.

1. Introduction

When sea waves impinge on a rigid obstacle, such as a breakwater, diffraction effects are produced similar to those found when a beam of light is partly cut off by an opaque screen. Consequently, when a reasonably exact picture is required of the shelter provided by a given arrangement of breakwaters, these diffraction effects must be taken into account. The mathematical theory of the diffraction of sea waves does not seem to have received much attention,* in spite of the very considerable studies of surface waves in classical hydrodynamics, but fortunately this theory has many points of similarity with the well-known theories of the diffraction of light waves and sound waves, and some of the mathematical problems solved in those theories can be suitably modified and developed to give useful information about sea waves.

We first show (§ 2) that the general distribution of surface waves on a sheet of water of any uniform depth, and their amplitude and phase at any point, are conveniently expressed in terms of a complex function $F(x,y)$, which we term the wave-function. The differential equation and boundary conditions satisfied by $F(x,y)$ are, in certain cases, identical with those satisfied by a corresponding function in the theory of the diffraction of light. Hence by making use of Sommerfeld's (1896) solution (see also Baker & Copson (1939)) for the diffraction of light waves at the edge of a semi-infinite screen, we determine the wave-function $F(x,y)$ for the diffraction of sea waves round the end of a long, straight breakwater. The wave

* Diffraction of sea waves by obstacles of narrow breadth has been treated by Havelock (1940) in discussing the wave resistance of ships; his results are not generally applicable to the problems we consider.

pattern and the distribution of wave height on the leeward side are deduced both when the incident waves are normal (§ 3) and oblique (§ 4) to the breakwater. These results are also used (§ 5) to obtain some idea of the conditions obtaining in the lee of a small island.

The mathematical theory of the diffraction of waves through a single gap in a breakwater is developed in §§ 6 to 8; in this case, the general nature of the result is found to depend on whether the gap is small or not compared with the length of the waves.

The assumption is made throughout part I that the height of the waves is small compared with their length, and consequently the wave-profile is nearly sinusoidal, but it seems probable that the results will not be altered seriously if the waves are of moderate height. The results will apply to water of any *uniform* depth; if the depth is not uniform important refraction effects may occur; for example, the waves will tend to turn so that their direction of motion is along the line of greatest slope of the bottom, because the velocity of the waves decreases with decreasing depth. Such effects would be important, and must be allowed for, in water of moderate depth, but are not dealt with in the present paper.

With actual breakwaters the incident waves are partly reflected and partly destroyed by turbulent motion. The present calculations deal only with ideal breakwaters in which the waves are completely reflected. From a purely theoretical point of view it is convenient to consider two types of ideal breakwaters, namely, (i) rigid breakwaters and (ii) flexible or 'cushion' breakwaters, as defined by equations (12*a*) and (12*b*) in § 3. Of these only (i) is probably of any practical importance, though it is of interest to remark that some examples of (ii) were designed, constructed and tested during the war, and were found to behave in the manner predicted by theoretical considerations.

2. Fundamental equations

Taking x- and y-axes in the plane of the undisturbed surface of the water, and the axis of z vertically upwards, we denote by ζ the elevation at time t of the surface at the point (x, y) above the undisturbed level. The motion of the water can be derived (Lamb 1924) from a velocity potential ϕ, satisfying the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad (z\leqslant 0). \tag{1}$$

When the wave height is not too great, the pressure p is given approximately by

$$p=\rho(\dot{\phi}-gz) \quad (z\leqslant 0), \tag{2}$$

and in order that p shall be constant at the surface $z=\zeta$, we require, to the first order of small quantities,

$$\zeta=\dot{\phi}/g \quad \text{at } z=0. \tag{3}$$

Also, since the normal component of the fluid velocity at the surface must equal the normal component of the velocity of the surface itself, we have approximately

$$\dot{\zeta}=-\frac{\partial\phi}{\partial z} \quad \text{at } z=0, \tag{4}$$

whence, from (3) and (4)

$$\ddot{\phi}+g\frac{\partial\phi}{\partial z}=0 \quad \text{at } z=0. \tag{5}$$

Assuming the water to be of depth d, we have

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -d. \tag{6}$$

Solutions of (1) which are periodic in t and satisfy the condition (6) are of the form

$$\phi = A\,e^{ikct} \cosh k(z+d)\, F(x,y), \tag{7}$$

where $F(x,y)$ is any complex function of (x,y) satisfying

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + k^2 F = 0, \tag{8}$$

and it is to be understood that the real part of the expression on the right of (7) is to be taken.

From (5) we have

$$-k^2c^2 \cosh kd + gk \sinh kd = 0,$$

so that

$$c^2 = \frac{g}{k} \tanh kd. \tag{9}$$

The elevation ζ of the surface is obtained from (3) as

$$\zeta = \frac{Aikc}{g}\, e^{kict} \cosh kd\, F(x,y), \tag{10}$$

where the real part of the expression on the right is to be taken. If $F(x,y)$ is any complex function satisfying (8), it is evident that the amplitude and phase of the surface wave will be determined by the modulus and argument of $F(x,y)$.

To represent progressive waves travelling, say, in the direction of the positive y-axis, we may take the solution

$$F(x,y) = e^{-iky} \tag{11}$$

of equation (8), whence

$$\zeta = \frac{Aikc}{g}\, e^{ik(ct-y)} \cosh kd,$$

or taking the real part

$$\zeta = \frac{Akc}{g} \cosh kd \sin k(ct-y). \tag{11a}$$

This represents waves, of wave-length $\lambda = 2\pi/k$, period $2\pi/kc$ and amplitude

$$a = (Akc \cosh kd)/g,$$

travelling with the velocity c, given by equation (9).

3. Diffraction of waves at the end of a long breakwater

We assume that the breakwater extends along the x-axis from the origin to infinity. If the breakwater is 'rigid', the normal component of the fluid velocity must be zero there, so that

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = 0,\ x \geqslant 0. \tag{12a}$$

If the breakwater is of the 'cushion' type, the pressure remains constant and equal to the hydrostatic pressure at all depths, so that from (2)

$$\frac{\partial \phi}{\partial t} = 0 \quad \text{at } y = 0,\ x \geqslant 0. \tag{12b}$$

Although the cushion type of breakwater is probably of no importance from a practical point of view (cf. § 2), it is convenient to develop the theory of both types simultaneously for reasons which appear later.

We consider first the diffraction of waves which are incident normally on the breakwater, so that they are travelling in the direction of the y-axis, and are therefore given by equations (11) and (11 a).

To determine their diffraction, we have to find a solution of equation (8) which will reduce to (11) when x is large and *negative*, and will satisfy (12 a) or (12 b) as the case may be, at the breakwater.

In terms of $F(x,y)$ the conditions (12 a), (12 b) become respectively

$$\frac{\partial F}{\partial y} = 0 \quad \text{at } y = 0,\ x \geqslant 0, \tag{13a}$$

and

$$F = 0 \quad \text{at } y = 0,\ x \geqslant 0. \tag{13b}$$

The equations (8), (11), (13 a) and (13 b) are identical with the equations of Sommerfeld's problem of the diffraction of light by a semi-infinite perfectly reflecting screen, the conditions (13 a) and (13 b) corresponding to the cases where the light is (a) polarized in a plane parallel to the edge of the screen, and (b) polarized in a plane perpendicular to the edge. Sommerfeld's solution (Sommerfeld 1896; Baker & Copson 1939) of these equations may be reduced to the form

$$F(x,y) = \frac{1+i}{2}\left\{e^{-iky}\int_{-\infty}^{\sigma} e^{-\frac{1}{2}\pi i u^2}\,du \pm e^{iky}\int_{-\infty}^{\sigma'} e^{-\frac{1}{2}\pi i u^2}\,du\right\}. \tag{14}$$

The + or − sign corresponds to the condition (13 a) or (13 b) respectively, and

$$\sigma^2 = \frac{4}{\lambda}(r-y), \quad \sigma'^2 = \frac{4}{\lambda}(r+y), \quad \lambda = \frac{2\pi}{k}, \tag{15}$$

where $r = \sqrt{(x^2+y^2)}$ and the signs for σ and σ' depend on the position of (x,y) in the four quadrants as indicated below

	y	
(+, −)	0	(−, −)
(+, −)		(+, +)

x (16)

Denoting by H the ratio of the maximum height of the waves at any point (x,y) to the height of the incident waves, and the phase difference of the waves by ϵ, we have

$$H = \operatorname{mod} F(x,y), \tag{17}$$

$$\epsilon = \arg F(x,y). \tag{18}$$

The expression (14) for $F(x,y)$ can be evaluated by using tables of Fresnel's integrals,

$$\int_0^u \cos\tfrac{1}{2}\pi u^2\,du \quad \text{and} \quad \int_0^u \sin\tfrac{1}{2}\pi u^2\,du,$$

or by graphical methods using Cornu's spiral (Jahnke & Emde 1945). It is convenient to determine first the modulus and argument of the function $f(\sigma)$, where

$$f(\sigma) = \tfrac{1}{2}(1+i)\int_{-\infty}^{\sigma} e^{-\frac{1}{2}\pi i u^2}\,du. \tag{19}$$

The graphs of mod $f(\sigma)$ and arg $f(\sigma)$ for values of between -3 and $+3$ are shown in figure 1.

To obtain approximations to $f(\sigma)$ for numerically large values of σ, we find on integrating by parts that, when σ is *negative*,

$$f(\sigma) = \tfrac{1}{2}(1+i)\left[-\frac{1}{\pi\sigma}\exp\left\{-\frac{\pi i}{2}(1+\sigma^2)\right\}+O(\sigma^{-3})\right]$$

$$= -\frac{1}{\pi\sigma\sqrt{2}}\exp\left\{-\frac{\pi i}{4}(1+2\sigma^2)\right\} \tag{20}$$

approximately, for large negative values of σ. An approximation for large positive values of σ is then easily deduced from the relation

$$f(\sigma)+f(-\sigma) = 1, \tag{21}$$

which is true for all real values of σ. We also find

$$\left.\begin{aligned} f(\sigma) &\to 1 \quad \text{as } \sigma\to+\infty,\\ f(\sigma) &\to 0 \quad \text{as } \sigma\to-\infty,\\ f(\sigma) &= \tfrac{1}{2} \quad \text{at } \sigma = 0. \end{aligned}\right\} \tag{22}$$

Using (22), we find the limiting forms of the solution (14) for various limiting positions of the point (x, y) are as shown in (23) below.

$F\to\frac{1}{2}e^{-iky}$ as $y\to+\infty$, $x=0$

$F\to 0$ as $x\to+\infty$, $y>0$ lee of breakwater (23)

$F\to e^{iky}$ as $x\to-\infty$ for all y

$F\to\frac{1}{2}e^{-iky}\pm\frac{1}{2}e^{iky}$ as $y\to 0$, $x=0$

$F\to e^{-iky}\pm e^{iky}$ as $x\to+\infty$, $y<0$

$F\to e^{-iky}\pm\frac{1}{2}e^{iky}$ as $y\to-\infty$, $x=0$

Remembering that the elevation ζ is given by the real part of (10), and considering the value of F for large negative values of x, it is evident that this solution corresponds to waves of the form (11 *a*) travelling in the direction Oy and incident normally on the breakwater.

On the leeward side of the breakwater F tends to zero for large positive x, agreeing with the obvious fact that the screening becomes more and more effective as we move farther away from the open end. At points directly behind the open end $(x = 0)$, F tends to $\frac{1}{2}e^{-iky}$ for large y, indicating that the amplitude of the waves is there reduced to half that of the waves in the open sea.

On the windward side of the breakwater, F contains the term e^{iky}, corresponding to the reflected waves moving in the negative y-direction. These combine with the incident waves to form stationary waves. At points well removed from the open end of the breakwater (x large and positive), the amplitude of these stationary waves is twice that of the travelling waves in the open sea. For a rigid breakwater, $F \to e^{-iky} + e^{iky}$ as $x \to \infty$ for negative y, indicating that the breakwater ($y = 0$) is at an antinode of the oscillating water surface, i.e. the water-level varies by its maximum amount at the breakwater. On the other hand, for a cushion-type breakwater, $F \to e^{-iky} - e^{iky}$ as $x \to \infty$, showing that the breakwater is at a node, so that the water-level remains constant.

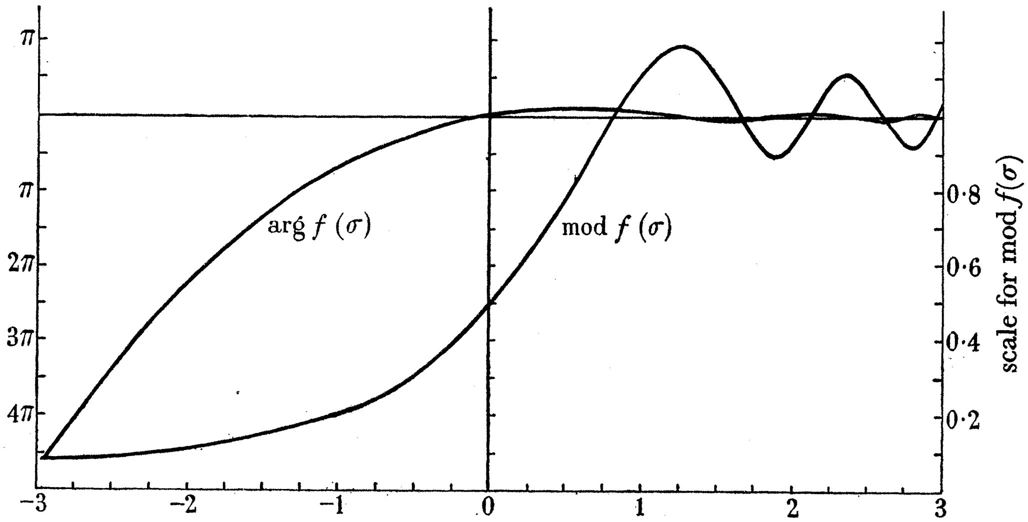

FIGURE 1. Graphs of the modulus and argument of the function $f(\sigma)$, where

$$f(\sigma) = \tfrac{1}{2}(1+i)\int_{-\infty}^{\sigma} e^{\frac{1}{2}i\pi u^2} du.$$

To determine the amplitude and phase of the waves at intermediate positions, we have to return to the general expression (14). On the lee side of the breakwater, for values of $y \geqslant 2$, the second integral in (14) will be small since $\sigma < -4$. Hence we have, very nearly,

$$F(x, y) = f(\sigma)\, e^{-i2\pi y/\lambda}, \tag{24}$$

where

$$\sigma = \mp\sqrt{\{4(r-y)/\lambda\}}, \tag{25}$$

the $-$ or $+$ sign being taken according as x is positive or negative. It follows from this that the conditions on the lee side of the breakwater at distances greater than about 2λ are nearly the same whether the breakwater is of the rigid or cushion type. Using (17) and (24), the maximum wave heights ($H = \operatorname{mod} F(x, y)$) have been determined as functions of x for distances $y = 2\lambda$ and 8λ behind the breakwater. These are shown in figure 2.

When y is large compared with x, we have

$$r - y = \sqrt{(x+y)^2} - y = x^2/2y \text{ approximately,}$$

so that the value of σ given by (25) is then approximately

$$\sigma = \mp\frac{x}{\lambda}\sqrt{\left(\frac{2\lambda}{y}\right)}. \tag{26}$$

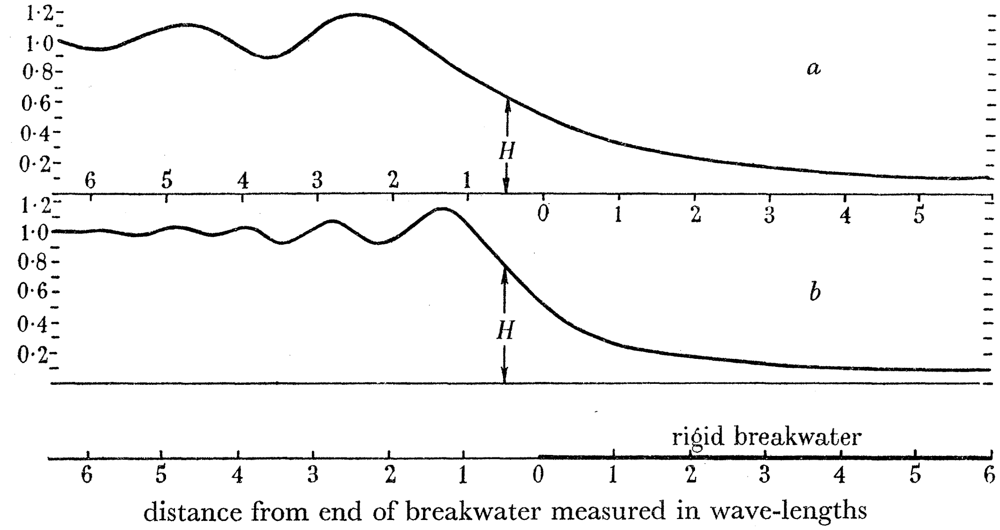

FIGURE 2. The maximum height of waves at normal incidence on a rigid breakwater, at distances $2\lambda\,(b)$ and $8\lambda\,(a)$ behind the breakwater. To obtain the wave heights at any distance $y = n\lambda$ (where $n > 4$) behind the breakwater, multiply the values at $y = 8\lambda$ by the factor $\sqrt{(\frac{1}{8}n)}$.

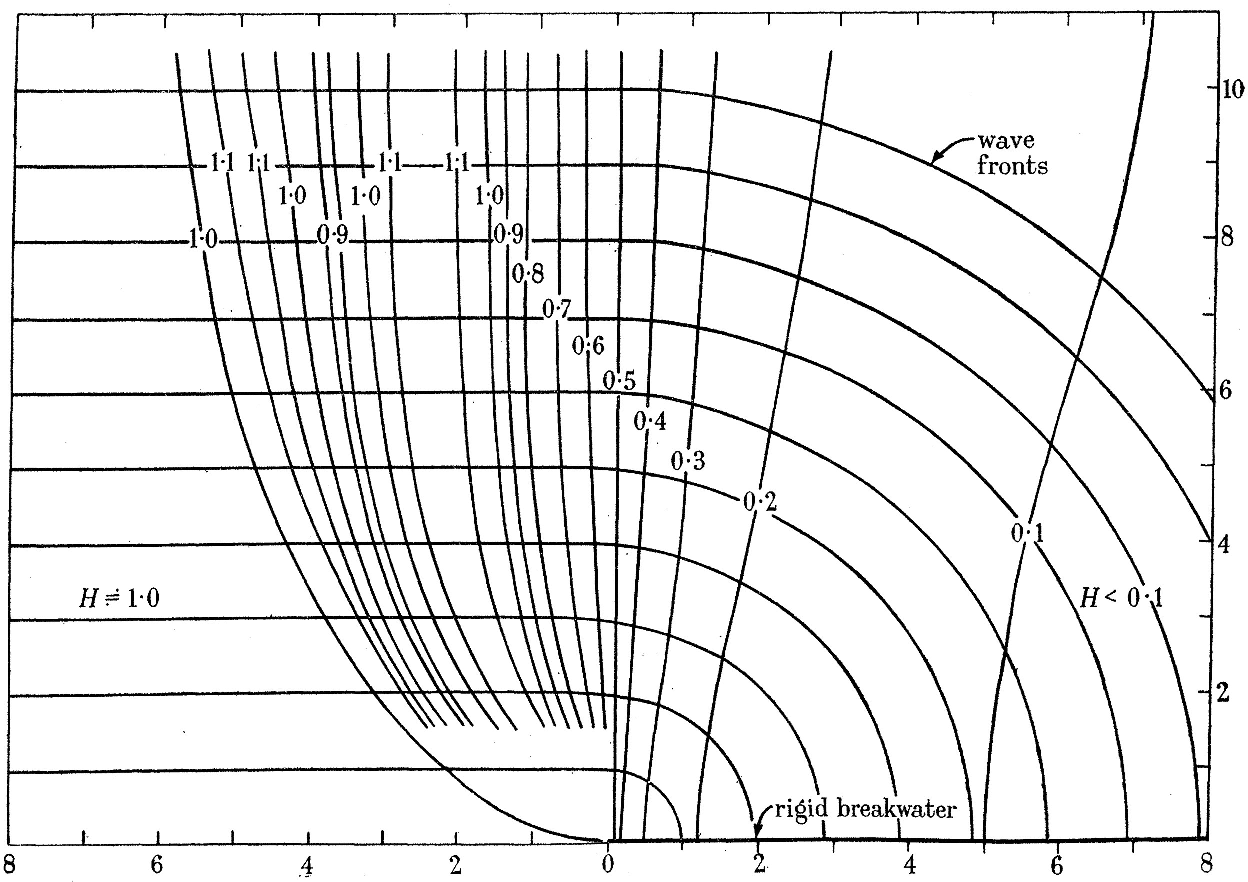

FIGURE 3. Wave fronts and contour lines of maximum wave heights in the lee of a rigid breakwater, the waves being incident normally.

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This shows that, for y large compared with x, the graph of the wave height as a function of x has the same form for different values of y, but the scale for the x-axis is proportional to $\sqrt{y}$. It follows that the graph of the wave height given in figure 2 for $y = 8\lambda$ will serve as the graph of the wave height for any distance $y = n\lambda$ from the breakwater, where n is greater than about 3, provided the numbers attached to the x-scale are multiplied by $\sqrt{(\frac{1}{8}n)}$.

Another method, which is particularly convenient, for exhibiting the distribution of maximum wave height H behind the breakwater is to draw the contour lines

$$\operatorname{mod} F(x,y) = H,$$

for a series of values of H. When $y \gg 2\lambda$, the approximation (24) may be used, so that

$$\operatorname{mod} F(x,y) = \operatorname{mod} f(\sigma).$$

Hence the value (or values) of σ, say σ_H, corresponding to any particular value of H may be read from the graph of $\operatorname{mod} f(\sigma)$ in figure 1. Equation (25) then shows that the corresponding contour line will be the portion of the parabola

$$4(r-y) = \lambda\sigma_H^2,$$

or in Cartesians,

$$\left(\frac{x}{\lambda}\right)^2 = \frac{\sigma_H^4}{16} + \frac{\sigma_H^2}{2}\left(\frac{y}{\lambda}\right), \tag{27}$$

for which x is positive when σ_H is negative, and x is negative when σ_H is positive. For points nearer to the breakwater, the contour lines will differ appreciably from the above parabolas, and their form will depend on whether the breakwater is of the rigid or cushion type. It is then necessary to calculate them from the exact expression (14). The contour lines for a rigid breakwater, corresponding to increments of 0·1 in H, are shown in figure 3. It will be observed that there are narrow bands extending behind the breakwater in the region beyond the open end ($x<0$) in which the wave height is slightly greater than in the open sea, but in the lee of the breakwater the wave height is always less than one-half that in the open sea.

To obtain the wave pattern, i.e. the shape of the wave fronts, behind the breakwater, we require the lines of constant phase. These are given by the curves $\arg F(x,y) = C$, where $F(x,y)$ is given approximately by (24) for $y > 2\lambda$. Beyond the open end of the breakwater (x negative) σ is positive, and it will be seen from the graph in figure 1 that $\arg f(\sigma)$ is then only slightly different from zero. Hence the wave fronts are, as we should expect, only slightly deformed from the straight-line fronts in the open sea. In the lee of the breakwater σ is negative and consequently $\arg f(\sigma)$ is negative, which indicates a phase-lag. This phase-lag increases with increasing x so that the wave fronts bend round towards the lee side of the breakwater. The wave fronts are shown in figure 3. It will be observed that in the lee of the breakwater, they are very nearly arcs of circles centred at the open end.

4. Waves incident obliquely on the breakwater

When the incident waves are travelling in a direction which makes an angle θ_0 with the breakwater, it is convenient to divide the surrounding area into three regions Q, R and S as in figure 4. The region S, where $0<\theta<\theta_0$, is in the main sheltered by the breakwater; the region Q, where $\theta_0<\theta<2\pi$, is relatively unaffected by the breakwater; and the region R, where $2\pi-\theta_0<\theta<2\pi$, contains waves reflected by the breakwater.

To obtain the wave heights and the wave pattern in these three regions, it is necessary to generalize slightly the preceding solution. In place of the expression (14) for $F(x, y)$, we now have, using polar co-ordinates (r, θ) instead of (x, y),

$$F(r,\theta) = \exp\{-ikr\cos(\theta-\theta_0)\}f(\sigma) \pm \exp\{-ikr\cos(\theta+\theta_0)\}f(\sigma'), \tag{28}$$

where $f(\sigma)$ is defined by (18), and

$$\sigma = 2\sqrt{\left(\frac{kr}{\pi}\right)}\sin\tfrac{1}{2}(\theta-\theta_0), \quad \sigma' = -2\sqrt{\left(\frac{kr}{\pi}\right)}\sin\tfrac{1}{2}(\theta+\theta_0), \tag{29}$$

with $0<\theta<2\pi$. As before, the + or − sign is to be taken in (28) according as the breakwater is of the rigid or flexible type.

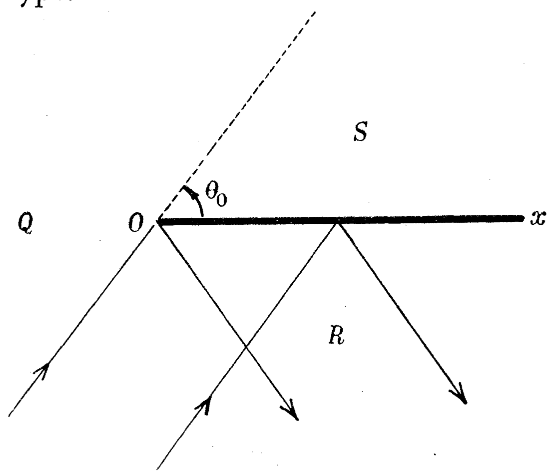

FIGURE 4. Notation for the regions Q, R, S for oblique incidence of waves on a breakwater.

It will be observed that, in the region Q, σ is positive and σ' is negative; in the region R both σ and σ' are positive; while in the region S both σ and σ' are negative. It will also be seen that, if θ_0 is put equal to $\frac{1}{2}\pi$ in (28) and (29), this solution reduces to the one previously given for normal incidence.

By making use of (21), the expression (28) can be analyzed to show separately the incident and reflected waves, and the diffraction effects. Thus in the region Q, since σ is positive there, we write $f(\sigma) = 1-f(-\sigma)$ in (28), giving

$$F(r,\theta) = \exp\{-ikr\cos(\theta-\theta_0)\} - \exp\{-ikr\cos(\theta-\theta_0)\}f(-\sigma) \pm \exp\{-ikr\cos(\theta+\theta_0)\}f(\sigma'). \tag{30}$$

The first term in this expression represents the incident waves, and the other two correspond to the diffraction effects. The latter terms tend to zero as r tends to infinity, because $-\sigma$ and σ' are each negative and proportional to $\sqrt{r}$ so that both $f(-\sigma)$ and $f(\sigma')$ tend to zero.

Similarly in the region R, since both σ and σ' are positive, we apply (21) to both $f(\sigma)$ and $f(\sigma')$ in (28), giving

$$F(r,\theta) = \exp\{-ikr\cos(\theta-\theta_0)\} \pm \exp\{-ikr\cos(\theta+\theta_0)\} - \exp\{-ikr\cos(\theta-\theta_0)\}f(-\sigma) \mp \exp\{-ikr\cos(\theta+\theta_0)\}f(-\sigma'). \tag{31}$$

In this expression the first two terms represent respectively the incident and reflected waves, and the last two terms represent the diffraction effects.

In the region S both σ and σ' are negative so that only diffraction effects, as represented by (28), are present.

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Thus the total effect of the breakwater can be regarded as the sum of (i) the shadow and reflexion effects as given by geometrical optics and represented by the simple exponential terms in (30) and (31), and (ii) the diffraction effects, which are given by the terms involving the function $f(u)$ where u is negative in (28), (30) and (31).

An approximate expression for the diffraction effects for large r (except when θ is nearly equal to θ_0 or $2\pi-\theta_0$) can be obtained by using the approximation (20). On substituting this in equation (28) and using (29), we find for the region S

$$F(r,\theta) = 0{\cdot}0796\sqrt{\left(\frac{\lambda}{r}\right)}\{\operatorname{cosec}\tfrac{1}{2}(\theta_0-\theta)\pm\operatorname{cosec}\tfrac{1}{2}(\theta_0+\theta)\}\exp\left\{-2\pi i\left(\frac{r}{\lambda}+\frac{1}{8}\right)\right\}. \quad (32)$$

For the regions Q and R, the last two terms in the expressions (30) and (31) lead to precisely the same expression as on the left of (32). Hence, except when θ is nearly equal to θ_0 or $2\pi-\theta_0$, the diffraction effects are represented at sufficiently great distances by circular waves diverging from the open end of the breakwater, as given by (32). The amplitude of these 'diffraction' waves decreases like $r^{-\frac{1}{2}}$ as r increases, except near the edge of the geometric shadow ($\theta=\theta_0$), or near the edge of the region of reflexion ($\theta=2\pi-\theta_0$).

Along the edge of the geometric shadow ($\theta=\theta_0$), the expression (28) for $F(r,\theta)$ reduces to approximately $\frac{1}{2}\exp(-2\pi ir/\lambda)$ for large r; this value is, in fact, a fair approximation for values of r as small as 2λ; for example, if $\theta_0 = 60^\circ$, we find

$$F(r,\theta) = 0{\cdot}50\,\mathrm{e}^{-4i\pi}\pm0{\cdot}07\,\mathrm{e}^{-(4\cdot2)i\pi} \quad \text{for } r=2\lambda, \quad (33)$$

the + or − sign corresponding, as before, to the breakwater being of the rigid or cushion type. Hence along the edge of the sheltered region S the height of the resultant waves is reduced to one-half that of the incident waves. Further inside S, each wave crest is bent round approximately into an arc of a circle centred at the open end of the breakwater, and its height is progressively reduced, in accordance with the expression (32), as the breakwater is approached. The expression (32) also shows that at points well inside S there is a phase lag of amount $\lambda/8$. This indicates that, near the boundary of S, the wave crest is bent round rather more than would be required to make it coincide with a circular arc centred at the open end. These features are illustrated in figures 5*a*, 5*b* and 6, which show the wave patterns and indicate approximately the wave heights by the widths of the shaded areas.

On moving out of S into the open region Q, there is a discontinuity of one-half wave-length in the phase of the *diffraction* waves, which are represented in S by (28), and in Q by the last two terms of (30) (cf. figure 5*b*); the amplitude is, however, continuous and equal approximately to 0·5 at the boundary. This has the effect of making the *resultant* wave fronts and wave heights continuous at the boundary, since near the boundary the diffraction waves in the Q region subtract from the incident waves, thereby reducing their height to one-half their normal height, and thus making them continuous with the waves inside the region S. On moving farther into the Q region, the height of the diffraction waves progressively decreases and becomes zero at $\theta=\pi$, i.e. along the continuation of the line of the breakwater. In the case of the rigid breakwater, but not for the cushion type, there is a change of phase again at $\theta=\pi$ of one-half wave-length, and as θ is further increased, the height increases again until it reaches the value 0·5 approximately at $\theta=2\pi-\theta_0$, i.e. at the boundary between the Q and R regions. At this boundary there is once again a phase change of one-half wave-length, the amplitude being continuous.

The effect of the diffraction waves just inside the R region is to decrease the height of the reflected waves near the $R-Q$ boundary to approximately one-half their normal height. These are superposed on the incident waves, so that the maximum height of the resultant waves near the boundary is $1\frac{1}{2}$ times that of the travelling waves in the open sea.

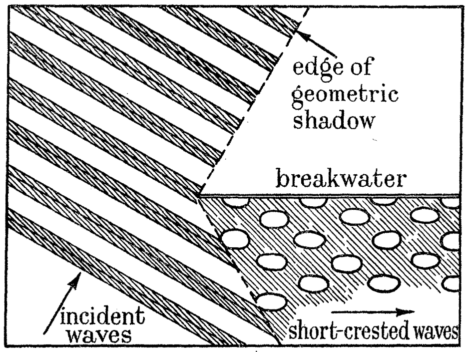

Figure 5a

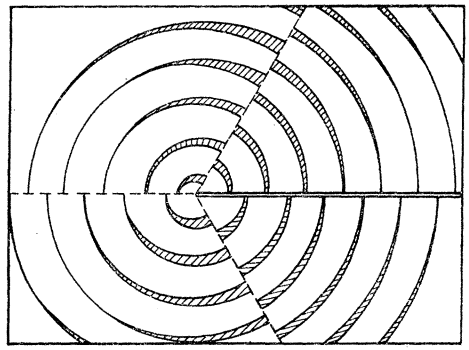

Figure 5b

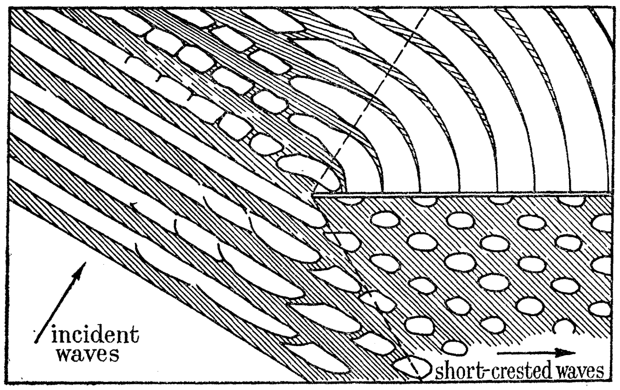

Figure 6

Figures 5a, b. Wave patterns for waves incident at 60° to a rigid breakwater. Figure 5a shows the pattern according to geometrical optics. Figure 5b shows the diffraction waves.

Figure 6. The wave pattern for waves incident at 60° to a rigid breakwater. The pattern shown is the superposition of the geometrical optics waves of figure 5a and the diffraction waves of figure 5b. The width of the shaded areas representing the waves gives an idea of their height.

Inside the R region, the reflected waves combine with the incident waves to form a system of short-crested waves, modulated both along and perpendicular to the breakwater, and travelling parallel to it. This may be seen by multiplying the first two terms of (31) by e^{ikct}, and taking the real part of the resulting expression. This gives for the combined waves

$$\zeta/\zeta_0 = 2\cos(y\sin\theta_0)\cos(ct-x\cos\theta_0) \tag{34}$$

if the breakwater is rigid, or

$$\zeta/\zeta_0 = 2\sin(y\sin\theta_0)\sin(ct-x\cos\theta_0) \tag{35}$$

if it is flexible, where ζ_0 is the amplitude of the incident waves, and x and y are measured parallel and perpendicular to the breakwater, with the origin at the open end. This shows that in both cases we have a system of short-crested waves, of length $2\pi/\cos\theta_0$ in the x-direction and length $2\pi/\sin\theta_0$ in the y-direction, travelling with velocity $c\sec\theta_0$ in the x-direction. It will be observed that the wave velocity becomes infinite as $\theta_0 \to \frac{1}{2}\pi$ (i.e. for normal incidence), but at the same time the length of the crests in the x-direction becomes infinite, so that in the limit we have simply a system of stationary waves modulated in the y-direction only.

These results are illustrated in figures 5*a*, 5*b* and 6, where the wave patterns are drawn for $\theta_0 = 60°$. Figure 5*a* shows the incident and reflected waves as given by geometrical optics; figure 5*b* shows the diffraction waves and figure 6 shows the resultant waves. The widths of the shaded areas representing the waves are a rough measure of their height.

5. The shelter afforded by an island breakwater

A rough idea of the conditions in the lee of a small steep-sided island, when the waves are coming from only one direction, may be obtained by treating it as a straight breakwater of finite length, equal to the width of the island in the direction of the wave-front, this length

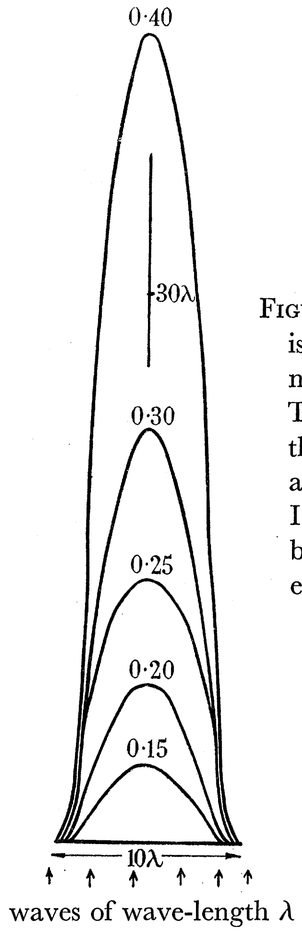

Figure 7. The wave pattern produced by an island of vertical sides, the waves being incident normally, and the island being of length 10λ. The contours shown are those of areas in which the wave height exceeds 0·15, 0·2, 0·25, 0·30 and 0·40 of the wave height in the open sea. It is assumed that there is no correlation between the diffraction patterns from the two ends of the island.

being a moderately large number (say 10 or more) of wave-lengths. The diffraction patterns at the two ends will be mirror images of each other, but the irregularities in the shore of the island and in the waves themselves will be sufficient to destroy correlation of these diffraction patterns. We can, however, deduce that the maximum wave height at any point will not exceed the sum of the maximum wave heights due to the diffraction round the two ends. These maximum heights will be given approximately by the wave-height distribution shown in figure 3 for one end, and by its mirror image for the other end. The distribution of wave height behind an island breakwater of effective width 10λ has been obtained in this way, and is illustrated in figure 7.

6. The transmission of waves through a single gap in a breakwater

We suppose now that there is a gap of breadth b, in a very long breakwater, on which waves are incident normally. If the gap is small, i.e. b small compared with the wave-length λ of the incident waves, the gap will behave like a point source, semicircular waves diverging from it and having the same height practically all round any semicircle. If, however, the gap is larger than a wave-length, the energy of the waves penetrating the gap will be propagated mainly in the original direction of the incident waves; the wave pattern at considerable distances from the gap will still be approximately semicircles, but the wave heights will vary along these semicircles, being greatest in the direction of propagation of the incident waves. It will be shown that, when b is greater than λ, a good approximation can be obtained by superposition from the solution for the semi-infinite barrier already considered. In fact this method gives an *exact* solution for *all* values of b when one of the barriers is of the rigid type and the other of the cushion type, and this solution is therefore given first, though it is unlikely to be of practical interest on its own account.

We take the origin at the centre of the gap, the x-axis along the breakwater and the y-axis in the direction of propagation of the incident waves. The problem now reduces to finding a function $F(x, y)$, satisfying equation (8) together with the boundary condition $\partial F/\partial y = 0$ *or* $F = 0$ at $y = 0$, $|x| > \frac{1}{2}b$, according as the breakwater is of the rigid or cushion type, and containing a term e^{-iky} for y negative, corresponding to the incident waves.

Now if we suppose for the moment that the part of the breakwater to the left of the gap (i.e. for $x < -\frac{1}{2}b$) is removed, we have the case of a semi-infinite barrier for which the solution is given by (14). It has been seen that this solution may be analyzed into two parts, one of which corresponds to the shadow and reflexion as given by geometrical optics, and the other corresponds to diffraction waves diverging from the end of the barrier. In fact, on making use of (19) and (21) as in § 4 we can write the expression (14) for $F(x, y)$ in the forms

$$F(x,y) = e^{-iky} - f_1 \pm g_1 \qquad \text{for } x < \tfrac{1}{2}b \text{ and for all } y, \tag{36}$$

$$= +f_1 \pm g_1 \qquad \text{for } x > \tfrac{1}{2}b \text{ and } y > 0, \tag{37}$$

$$= e^{-iky} \pm e^{+iky} - f_1 \mp g_1 \qquad \text{for } x > \tfrac{1}{2}b \text{ and } y < 0, \tag{38}$$

where

$$f_1 = \phi(r_1, y) = e^{-iky} f\left(-\sqrt{\left\{\frac{4}{\lambda}(r_1 - y)\right\}}\right), \tag{39}$$

$$g = \phi(r_1, -y), \tag{40}$$

and r_1 is the distance of the point (x, y) from the end of the barrier $(\frac{1}{2}b, 0)$.

The diffraction waves in the region $x < \frac{1}{2}b$, due to the barrier along the line segment $y = 0$, $x > \frac{1}{2}b$, are represented by the terms $-f_1 \pm g_1$ ($= F_1$ say) in the expression (36) for $F(x, y)$. From (39) and (40) we find that these diffraction waves satisfy the following conditions along the *unoccupied* part of the x-axis:

$$F_1 = 0, \quad \frac{\partial F_1}{\partial y} = -2\frac{\partial}{\partial y}\phi(r_1, y) \quad \text{at } y = 0,\ x < \tfrac{1}{2}b, \tag{41}$$

for a rigid breakwater, and

$$F = 2\phi(r_1, 0), \quad \frac{\partial F_1}{\partial y} = 0 \quad \text{at } y = 0,\ x < \tfrac{1}{2}b, \tag{42}$$

for a flexible breakwater.

Now if a second barrier is introduced along the line segment $y = 0$, $x < -\frac{1}{2}b$, the diffraction waves arising at the first barrier would satisfy boundary conditions at the second barrier, which are appropriate to a barrier of opposite type from the first. For example, if the first barrier is rigid, the diffraction waves would satisfy the condition $F_1 = 0$ at the second barrier, which is the condition for a cushion barrier. Hence, if this second barrier is of the cushion type, the diffraction waves from the first barrier will be unaffected by the presence of the second barrier. It will, however, reflect the original incident waves and set up a further set of diffraction waves which will satisfy the necessary boundary condition $\partial F/\partial y = 0$ at the first barrier.

Hence by superposing the diffraction waves for the two barriers we get the scheme of values for $F(x, y)$ shown in (43), which gives an *exact* solution for the problem when one barrier is of the rigid type and the other of the cushion type:

$-f_1+g_1$		f_1+g_1	
f_2-g_2		$-f_2-g_2$	
	e^{-iky}		
———————	$-f_1+g_1$	———————	
cushion barrier	$-f_2-g_2$	rigid barrier	(43)
$e^{-iky}-e^{-iky}$		$e^{-iky}+e^{-iky}$	
$-f_1+g_1$		$-f_1-g_1$	
$-f_2+g_2$		$-f_2-g_2$	

It will be noted that this solution is exact for *all* values of b however small. The particular case when $b = 0$ solves the interesting problem of a change from a rigid to a cushion reflector at $x = 0$. There is a difference of phase of half-wave-length in the reflected waves in the two regions, but the wave pattern is made continuous by the f and g terms; the wave height is zero along the line $x = 0$.

When both barriers are of the same type, the boundary conditions at each barrier are not automatically satisfied by the diffraction waves arising from the incident waves on the other barrier, so that a secondary system of diffraction waves would need to be introduced to obtain an exact solution. It will be shown, however, that these secondary waves are relatively unimportant if b is greater than one wave-length. Thus, when both barriers are rigid breakwaters, we consider the scheme of values for $F(x, y)$ given in (44):

$-f_1+g_1$		f_1+g_1	
$+f_2+g_2$		$-f_2+g_2$	
	e^{-iky}		
———————	$-f_1+g_1$	———————	
rigid breakwater	$-f_2+g_2$	rigid breakwater	(44)
$e^{-iky}+e^{iky}$		$e^{-iky}+e^{iky}$	
$-f_1+g_1$		$-f_1-g_1$	
$-f_2-g_2$		$-f_2+g_2$	

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The boundary condition at either barrier should now be $\partial F/\partial y = 0$, but we find, for example, at the left-hand barrier, that all the terms in $\partial F/\partial y$ cancel out, except the term $\partial F_1/\partial y$, as given by (41). Using (39) and (19), we find that

$$\frac{\partial F}{\partial y} = \frac{\partial F_1}{\partial y} = \frac{4\pi i}{\lambda} f\left(-2\sqrt{\frac{r_1}{\lambda}}\right) - \sqrt{\left(\frac{2}{\lambda r_1}\right)} \exp\left(\frac{\pi i}{4} - \frac{2\pi i r_1}{\lambda}\right),$$

and, to the degree of approximation given by (20), this is equal to

$$\frac{2i}{\sqrt{(2\lambda r_1)}} \exp\left(-\frac{\pi i}{4} - \frac{2\pi i r_1}{\lambda}\right) - \frac{2}{\sqrt{(2\lambda r_1)}} \exp\left(\frac{\pi i}{4} - \frac{2\pi i r_1}{\lambda}\right) = 0.$$

Now (20) is correct to within 2 % of the maximum value of its modulus if $\sigma < -2$; hence the above condition is satisfied to this degree of approximation if $r_1 > \lambda$. The minimum value of r_1 is the width b of the gap. We can therefore conclude that (44) is a good approximation to the solution for two rigid breakwaters if the gap is not less than one wave-length in width.

7. The wave pattern and wave height behind the gap

The wave pattern will be given by the lines $\arg F(x, y)$ = constant and the wave height as compared with that of the incident waves by $\mathrm{mod}\, F(x, y)$. The nature of the function $F(x, y)$ can be inferred from the equation (39) for $\phi(r, y)$, which determines the functions f_1, f_2, g_1 and g_2. The modulus and argument of $\phi(r, y)$ can be obtained from the graphs in figure 1 for values of $r \pm y$ up to $2{\cdot}25\lambda$, σ being negative, and for larger values we can use the approximation (20).

When $r \mp y$ is zero, we have $\phi(r, \pm y) = \frac{1}{2} e^{-iky}$. Hence along the edge $(x = \frac{1}{2}b, y > 0)$ of one of the sheltered regions we have $f_1 = \frac{1}{2} e^{-iky}$, and along the edge $(x = -\frac{1}{2}b, y > 0)$ of the other sheltered region we have $f_2 = \frac{1}{2} e^{-iky}$. These are the largest values f_1 and f_2 can have, for it will be seen from the graph of $f(\sigma)$ that $\mathrm{mod}\, \phi(r, \pm y)$ is $< \frac{1}{2}$ at all points, since the corresponding value of σ is always negative. Moreover, $\mathrm{mod}\, \phi(r, \pm y)$ decreases fairly quickly at first as $(-\sigma)$ increases, i.e. as $r \pm y$ increases. For example, it is reduced to 0·08 when $\sigma = -2\sqrt{2}$, i.e. when $r \mp y = 2\lambda$. For larger values of $r \mp y$ we have the approximate expression

$$\phi(r, \pm y) = \sqrt{\left\{\frac{\lambda}{8\pi^2 (r \mp y)}\right\}} \exp\left(-\frac{i\pi}{4} - \frac{2i\pi r}{\lambda}\right). \tag{45}$$

One immediate deduction is that, on the lee side of the breakwater $(y > 0)$ the functions g_1 and g_2 will be small except for points within one or two wave-lengths from either end of the gap, and we may therefore ignore g_1 and g_2 compared with the other terms in $F(x, y)$. This means that the wave pattern and the wave height *behind the breakwater* are practically the same for all the three cases, viz. both barriers rigid, both of the cushion type, or one of each type.

At points opposite the gap $(-\frac{1}{2}b < x < \frac{1}{2}b)$, we have

$$F(x, y) = e^{-iky} - f_1 - f_2.$$

If the gap is broad, f_1 and f_2 are small except at $x = \pm\frac{1}{2}b$, where one of them is $\frac{1}{2} e^{-iky}$ and the other is small. This means, of course, that the incident waves pass through the gap almost unchanged except near the edges of the gap, where their height is reduced by one-half. When the gap is narrower, of the order of one or two wave-lengths, we can obtain an approximate expression for $F(x, y)$ as follows.

Since $r_1^2 = y^2+(x-\frac{1}{2}b)^2$, we have, when y is large compared with $(x-\frac{1}{2}b)$,

$$r_1-y = \tfrac{1}{2}(x-\tfrac{1}{2}b)^2/y \quad \text{approximately,}$$

and similarly

$$r_2-y = \tfrac{1}{2}(x+\tfrac{1}{2}b)^2/y \quad \text{approximately.}$$

Now from (39) and (19) we have

$$\phi(r,y) = \tfrac{1}{2}\,\mathrm{e}^{-iky}\,(1+i)\left\{\tfrac{1}{2}(1-i)+\int_0^{-w}\mathrm{e}^{-\frac{1}{2}\pi i u^2}du\right\},$$

where $w = \surd\{4(r-y)/\lambda\}$, and for small values of w we can expand the integrand as a power series, giving

$$\phi(r,y) = \tfrac{1}{2}\,\mathrm{e}^{-iky}\left\{1-w-\frac{\pi w^3}{6}\ldots+i\left(-w+\frac{\pi w^3}{6}\ldots\right)\right\}. \tag{46}$$

Hence, using (46), we find

$$F(x,y) = \mathrm{e}^{-iky}-f_1-f_2$$

$$= \mathrm{e}^{-iky}\frac{b}{\surd(\lambda y)}\left\{\mathrm{e}^{\frac{1}{4}\pi i}+\frac{\pi}{3\lambda y}\,(3x^2+\tfrac{1}{4}b^2)\,\mathrm{e}^{-\frac{1}{4}\pi i}\right\}. \tag{47}$$

This gives

$$\operatorname{mod} F(x,y) = \frac{b}{\surd(\lambda y)}\left\{1+\frac{\pi^2}{18\lambda^2y^2}\,(3x^2+\tfrac{1}{4}b^2)\right\}, \tag{48}$$

$$\arg F(x,y) = \tan^{-1}\left\{\frac{3\lambda y-\pi(3x^2+\frac{1}{4}b^2)}{3\lambda y+\pi(3x^2+\frac{1}{4}b^2)}\right\}-\frac{2\pi y}{\lambda}. \tag{49}$$

The above approximation holds good for points behind the gap at distances large compared with $\frac{1}{2}b-|x|$. The same approximation is found for points just inside the sheltered regions, on using $F(x,y) = f_1 \sim f_2$ and taking $|x|-\frac{1}{2}b$ small compared with y.

The expression (49) shows that $\arg F(x,y)$ lies between $-ky$ and $\frac{1}{4}\pi-ky$ for the whole range of values of x considered. Hence the waves in the region $-\frac{1}{2}b<x<\frac{1}{2}b$ remain almost parallel to the original incident waves and penetrate to some extent into the sheltered regions. But at points well within the shelter of either barrier, the functions f_1 and f_2 may be calculated from the expression (45) for $\phi(r, \pm y)$, which shows that the waves eventually curve round into a circular form, the waves corresponding to the separate terms f_1 and f_2 diverging from the ends $x = \frac{1}{2}b$ and $x = -\frac{1}{2}b$ of the gap.

The expression (48) for $\operatorname{mod} F(x,y)$ shows that the waves are nearly of uniform height $b/\surd(\lambda y)$ in the range $-\frac{1}{2}b<x<\frac{1}{2}b$, though there is a slight minimum at $x = 0$. But it will be noted that the second term on the right of (48) is very small for all admissible values of x, so that the increase in height on moving outwards from $x = 0$ is also small. This expression for the height will hold for a short distance into the sheltered regions, but at points farther in, the functions f_1 and f_2 rapidly decrease, so that the height of the waves correspondingly decreases as the waves curve inwards towards the breakwater.

The heights of the waves and their phase differences at various points for the case when $b = 2{\cdot}5\lambda$ are shown in table 1.

TABLE 1. WAVE HEIGHTS AND PHASE DIFFERENCES AT POINTS BEHIND A BREAKWATER WITH A GAP OF WIDTH $2{\cdot}5\lambda$

X = distance from line through mid-point of gap in wave-lengths.
Y = distance behind gap measured in wave-lengths.

Y \ X	0	0·5	1·0	1·25	1·5	2·0	4	8	12	16	20	24
5	1·01	0·92	0·73	0·57	0·42	0·29	0·21	0·04	—	—	—	—
	28°	24°	2°	−13°	−30°	−180°	−304°	−1975°				
10	0·77	0·76	0·69	0·66	0·62	0·52	0·10	0·13	0·03	—	—	—
	36°	33°	22°	12°	−1°	−26°	−185°	−798°				
25	0·51	0·50	0·49	0·47	0·44	0·43	0·39	0·09	0·05	—	—	—
	41°	40°	40°	23°	25°	10°	−60°	−405°				
50	0·35	0·35	0·35	0·35	0·32	0·31	0·34	0·28	0·19	0·05	0·01	—
	43°	42°	40°	37°	35°	20°	−10°	−140°	−328°			
100	0·25	0·25	0·25	0·26	0·25	0·24	0·23	0·20	0·23	0·20	0·16	0·10
	44°	42°	40°	39°	39°	37°	23°	−60°	−207°	−415°	−675°	−1040°

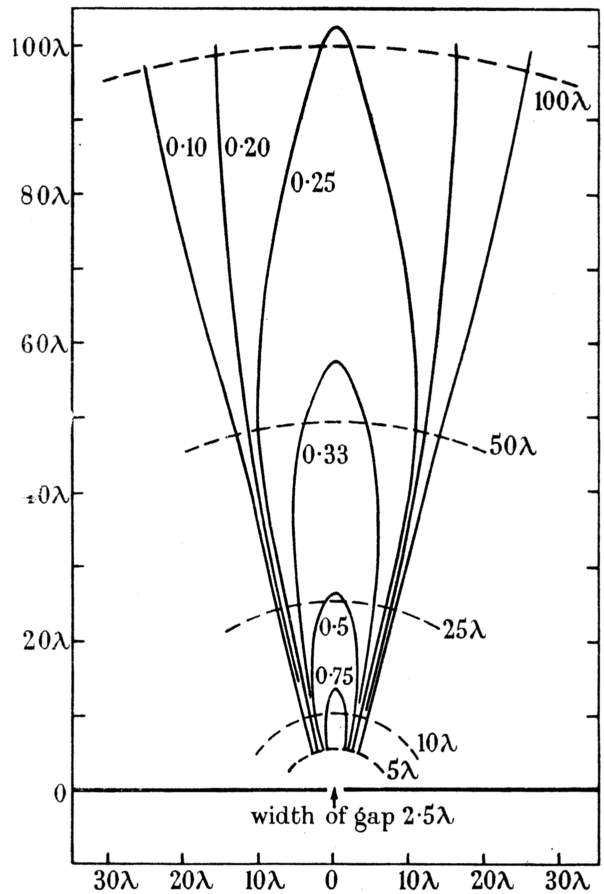

FIGURE 8. Wave heights behind a gap of width $2{\cdot}5\lambda$ in a rigid breakwater, the waves being incident normally.

The phase angles given in table 1 show the difference in phase between the incident waves at the point (if they had not been disturbed by the breakwater) and the actual waves at the point. The increasing negative value of these phase differences for increasing x indicates that the waves are bent backwards behind the breakwater, a phase difference of 360° corresponding, of course, to one wave-length. The wave pattern and distribution of wave height behind the breakwater are shown in figure 8.

8. Penetration of waves through a gap smaller than one wave-length

When b is small compared with λ, an approximate solution may be obtained by assuming that the stream-lines for the motion of the water through the gap are the same as for a simple *uniform* streaming of water through the gap. The mathematical problem then becomes identical with the problem of the penetration of sound waves through a narrow slit; this problem is treated by Lamb (1924). Using Lamb's solution we deduce the following expression for $F(x, y)$ for points not too near the gap:

$$F = \frac{(\lambda/4r)\exp(-\frac{1}{4}\pi i - kri)}{\log(\pi b/4\lambda) + \gamma + \frac{1}{2}\pi i}, \tag{50}$$

where r is the distance from the centre of the gap, and γ is Euler's constant, equal to 0·5772.

This corresponds to waves diverging uniformly in all directions behind the breakwater from the gap as a point source, the heights of the wave at any point being given by mod F, which is proportional to $r^{-\frac{1}{2}}$.

The expression for the height of the waves (mod F) may be written in the form

$$\operatorname{mod} F = \frac{\pi}{2\sqrt{(kb\{\log(\frac{1}{8}kb+\gamma)^2+\frac{1}{4}\pi^2\})}}\sqrt{\left(\frac{b}{\pi r}\right)}, \tag{51}$$

and the coefficient of $\sqrt{(b/\pi r)}$ in this expression is slightly greater than unity when b lies between $0{\cdot}1\lambda$ and $0{\cdot}2\lambda$. Thus when $b = \lambda/2\pi$ the value of the coefficient is 1·053. Hence for this range of values of b, the height of the waves is given approximately by $\sqrt{(b/\pi r)}$. This may be compared with the approximate formula $b/\sqrt{(\lambda y)}$ for the maximum height when $b > \lambda$. These two expressions would be equal at $r = y$ if $b^2/\lambda y = b/\pi r$, i.e. $b = \lambda/\pi$, which suggests that the formula $b/\sqrt{(\lambda y)}$ for the maximum wave height gives a rough approximation for values of b down to about λ/π. The fact that the maximum height is proportional to b and not $\sqrt{b}$ when the gap is not very small means that the energy of the wave motion passing through the gap is still propagated mainly in the direction of the incident waves.

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